

# Computing fields near RWG panels in SCUFF-EM

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In this note I discuss the evaluation of the  $\mathbf{E}$  and  $\mathbf{H}$  fields due to RWG currents on a single triangular source panel  $\mathcal{P}$  in the case where the evaluation point lies on or near the source panel. My method is essentially that of Graglia<sup>1</sup>; this is a desingularization scheme in which the first few terms in the low-frequency series expansion of the Green's function are subtracted off and evaluated analytically. However, in SCUFF-EM I need also to compute the *derivatives* of the  $\mathbf{E}$  and  $\mathbf{H}$  fields, which requires an extension of Graglia's methods.

The algorithm described here is implemented in the file `GetNearFields.cc` in the SCUFF-EM source distribution.

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<sup>1</sup>Graglia, *IEEE Trans. Ant. Prop.* **41** 1448 (1993); see also Wilton et al., *IEEE Trans. Ant. Prop.* **32** 276 (1984).

# 1 Fields from reduced potentials

## 1.1 Reduced vector and scalar potentials

Consider a triangular panel  $\mathcal{P}$  on which we have a flow of surface current described by an RWG basis function  $\mathbf{b}_\alpha(\mathbf{x})$  with source vertex  $\mathbf{Q}$ . I begin by defining “reduced” vector and scalar potentials produced by this current at a point  $\mathbf{x}$ :

$$\mathbf{a}(\mathcal{P}; \mathbf{Q}; \mathbf{x}) = \frac{\ell}{2A} \int_{\mathcal{P}} (\mathbf{x}' - \mathbf{Q}) G(\mathbf{x}, \mathbf{x}') d\mathbf{x}', \quad p(\mathcal{P}; \mathbf{x}) = 2 \cdot \frac{\ell}{2A} \int_{\mathcal{P}} G(\mathbf{x} - \mathbf{x}') d\mathbf{x}', \quad (1)$$

where  $\ell$  is the length of the edge associated with basis function  $\mathbf{b}$ ,  $A$  is the area of  $\mathcal{P}$ , and

$$G(\mathbf{r}) = \frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|}$$

is the scalar Helmholtz Green’s function.

The total reduced vector and scalar potentials of  $\mathbf{b}_\alpha$  involve contributions from two panels:

$$\mathbf{a}_\alpha(\mathbf{x}) = \mathbf{a}(\mathcal{P}^+, \mathbf{Q}^+, \mathbf{x}) - \mathbf{a}(\mathcal{P}^-, \mathbf{Q}^-, \mathbf{x}), \quad p_\alpha(\mathbf{x}) = p(\mathcal{P}^+, \mathbf{Q}^+, \mathbf{x}) - p(\mathcal{P}^-, \mathbf{Q}^-, \mathbf{x}).$$

In terms of  $\mathbf{a}_\alpha$  and  $p_\alpha$ , I can define the “reduced fields” of basis function  $\mathbf{b}_\alpha$  according to

$$\mathbf{e}_\alpha(\mathbf{x}) = \mathbf{a}_\alpha(\mathbf{x}) + \frac{1}{k^2} \nabla p_\alpha(\mathbf{x}), \quad \mathbf{h}_\alpha(\mathbf{x}) = \nabla \times \mathbf{a}_\alpha(\mathbf{x}) \quad (2)$$

Given a set of RWG functions  $\{\mathbf{b}_\alpha\}$  populated with electric and magnetic surface-current coefficients  $\{k_\alpha, n_\alpha\}$ , the full  $\mathbf{E}$  and  $\mathbf{H}$  fields at  $\mathbf{x}$  are

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \sum_{\alpha} \left\{ ikZk_{\alpha} \mathbf{e}_{\alpha}(\mathbf{x}) - n_{\alpha} \mathbf{h}_{\alpha}(\mathbf{x}) \right\}, \\ \mathbf{H}(\mathbf{x}) &= \sum_{\alpha} \left\{ k_{\alpha} \mathbf{h}_{\alpha}(\mathbf{x}) + \frac{ik}{Z} n_{\alpha} \mathbf{e}_{\alpha}(\mathbf{x}) \right\} \end{aligned}$$

where  $k$  is the wavenumber in the material region containing  $\mathbf{x}$  and  $Z = Z_0 Z^r$  with  $Z_0$  the impedance of free space and  $Z^r$  the relative wave impedance of the material.

To compute the first derivatives of the reduced fields we need the second derivatives of the reduced potentials:

$$\partial_i e_j(\mathcal{P}^{\pm}; \mathbf{x}) = \partial_i a_j(\mathcal{P}^{\pm}; \mathbf{x}) + \frac{1}{k^2} \partial_i \partial_j p(\mathcal{P}^{\pm}, \mathbf{c}), \quad \partial_i h_j(\mathcal{P}^{\pm}; \mathbf{x}) = \varepsilon_{jkl} \partial_i \partial_k a_l(\mathcal{P}^{\pm}; \mathbf{x}).$$

## 1.2 Desingularized reduced potentials

To compute  $\mathbf{a}$  and  $p$  at evaluation points  $\mathbf{x}$  on or near the source panel  $\mathcal{P}$  it is convenient to invoke the expansion

$$G(r) = \frac{e^{ikr}}{4\pi r} = \frac{1}{4\pi r} + (ik) \frac{1}{4\pi} + (ik)^2 \frac{r}{8\pi} + \frac{\text{ExpRelBar}(ikr, 3)}{4\pi r}$$

where `ExpRelBar` is just the usual exponential minus the first few terms in its series expansion:<sup>2</sup>

$$\text{ExpRelBar}(x, N) = e^x - \sum_{n=0}^{N-1} \frac{x^n}{n!} = \sum_{n=N}^{\infty} \frac{x^n}{n!}.$$

The reduced potentials due to  $\mathcal{P}$  are

$$p(\mathbf{x}) = 2 \frac{\ell}{2A} \left[ \sum_{n=-1}^1 \frac{(ik)^{n+1}}{4\pi} p^{(n)}(\mathbf{x}) + p^{\text{DS}}(\mathbf{x}) \right]$$

$$\mathbf{a}(\mathbf{x}) = \frac{\ell}{2A} \left[ \sum_{n=-1}^1 \frac{(ik)^{n+1}}{4\pi} \mathbf{a}^{(n)}(\mathbf{x}) + \mathbf{a}^{\text{DS}}(\mathbf{x}) \right]$$

where<sup>3</sup>

$$p^{(n)}(\mathbf{x}) = \int_{\mathcal{P}} r^n dA \quad (3)$$

$$\mathbf{a}^{(n)}(\mathbf{x}) = \int_{\mathcal{P}} (\mathbf{x}' - \mathbf{Q}) r^n dA. \quad (4)$$

(here  $r = |\mathbf{x}' - \mathbf{x}|$ ). To write  $\mathbf{a}^{(n)}$  in a more convenient form, let  $\bar{\mathbf{x}}$  be the projection of the evaluation point into the plane of the panel and write

$$\begin{aligned} \mathbf{a}^{(n)}(\mathbf{x}) &= \int_{\mathcal{P}} (\mathbf{x}' - \bar{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{Q}) r^n dA \\ &= \int_{\mathcal{P}} (\mathbf{x}' - \bar{\mathbf{x}}) r^n dA - (\mathbf{Q} - \bar{\mathbf{x}}) \int_{\mathcal{P}} r^n dA. \end{aligned}$$

To proceed I now define scalar and vector-valued functions according to<sup>4</sup>

$$\mathcal{I}^p(\mathcal{P}, \mathbf{x}) \equiv \int_{\mathcal{P}} r^p dA \quad (5a)$$

$$\mathcal{J}^p(\mathcal{P}, \mathbf{x}) \equiv \int_{\mathcal{P}} \bar{\mathbf{r}} \cdot r^p dA. \quad (5b)$$

<sup>2</sup>`ExpRelBar` is similar to, but distinct from, the function named `ExpRel` in the GNU SCIENTIFIC LIBRARY.

<sup>3</sup>Two special cases that may be computed analytically are  $p^{(0)} = A$ ,  $\mathbf{a}^{(0)} = A(\mathbf{x}_c - \mathbf{Q})$  where  $A$  is the panel area and  $\mathbf{x}_c$  is its centroid. These are independent of the evaluation point  $\mathbf{x}$  and thus do not contribute to derivatives.

<sup>4</sup>Note that I use a  $p$  superscript for  $\mathcal{I}$  and  $\mathcal{J}$  but an  $(n)$  superscript for  $\mathbf{a}$  and  $p$ . This is because the indices  $p$  and  $n$  have different ranges: The  $n$  values for which I need  $\mathbf{a}^{(n)}$  and  $p^{(n)}$  and their derivatives are  $n = -1, 0, 1$ , but computing all of these quantities turns out to require  $\mathcal{I}^p$  and  $\mathcal{J}^p$  for  $p = -5, -3, -1, 1$ .

where  $\bar{\mathbf{r}} = \mathbf{x}' - \bar{\mathbf{x}}$  is a vector that has nonzero components only in the plane of  $\mathcal{P}$ . In terms of  $\mathcal{I}$  and  $\mathcal{J}$ , the reduced potentials are

$$p^{(n)}(\mathcal{P}; \mathbf{x}) = \mathcal{I}^n(\mathcal{P}; \mathbf{x}), \quad \mathbf{a}^{(n)}(\mathcal{P}; \mathbf{x}) = \mathcal{J}^n(\mathcal{P}; \mathbf{x}) - (\mathbf{Q} - \bar{\mathbf{x}})\mathcal{I}^n(\mathcal{P}; \mathbf{x}) \quad (6)$$

Similarly, derivatives of  $\mathbf{a}^{(n)}$  and  $p^{(n)}$  are related to derivatives of  $\mathcal{I}^p$  and  $\mathcal{J}^p$ :

$$\begin{aligned} \partial_i p^p(\mathbf{x}) &= d_i \mathcal{I}^p(\mathcal{P}, \mathbf{x}) \\ \partial_i a_j^p(\mathbf{x}) &= \partial_i \mathcal{J}_j^p(\mathcal{P}, \mathbf{x}) - (\mathbf{Q} - \bar{\mathbf{x}})_j \partial_i \mathcal{I}^p - \delta_{ij} \mathcal{I}^p(\mathcal{P}, \mathbf{x}) \end{aligned}$$

The two-dimensional integrals in the quantities  $\mathcal{I}^p$  and  $\mathcal{J}^p$  defined by Equation (5b), as well as their derivatives, may be evaluated analytically in closed form, as discussed in the following section.

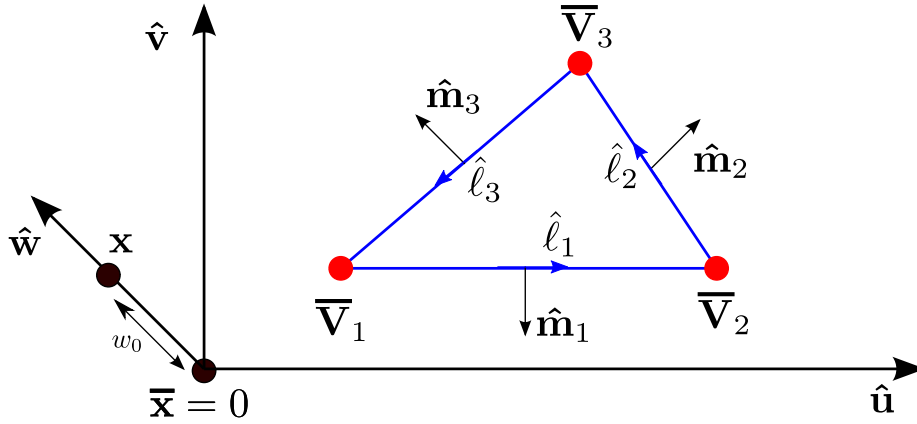


Figure 1: Rotated and translated coordinate system  $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}})$  for evaluation of  $\mathcal{I}, \mathcal{J}$  integrals. The panel lies in the  $uv$  plane with one edge parallel to the  $\hat{\mathbf{u}}$  axis. The origin of the  $uv$  plane is the projection of the evaluation point into the plane of the panel. The perpendicular distance from the plane of the panel to the evaluation point is  $w_0$ .

## 2 Evaluation of $\mathcal{I}, \mathcal{J}$ integrals

### 2.1 Modified coordinate system

Let the panel  $\mathcal{P}$  have vertices  $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ . For convenience in what follows, we introduce a rotated and translated coordinate system with cartesian coordinates  $(u, v, w)$ , in which  $\mathcal{P}$  lies entirely in the  $uv$  plane (the panel normal  $\hat{\mathbf{n}}$  defines the  $\hat{\mathbf{w}}$  axis) and edge  $\mathbf{V}_1\mathbf{V}_2$  lies parallel to the  $\hat{\mathbf{u}}$  axis (Figure 1). The unit vectors of this system are

$$\hat{\mathbf{u}} = \frac{\mathbf{V}_2 - \mathbf{V}_1}{|\mathbf{V}_2 - \mathbf{V}_1|}, \quad \hat{\mathbf{v}} = \hat{\mathbf{w}} \times \hat{\mathbf{u}}, \quad \hat{\mathbf{w}} = \frac{(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)}{|(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)|}$$

In the modified coordinate system, the evaluation point has coordinates  $\mathbf{x} = (\boldsymbol{\rho}_0, w_0)$  with

$$w_0 = (\mathbf{x}_0 - \mathbf{x}_c) \cdot \hat{\mathbf{w}}, \quad \boldsymbol{\rho}_0 = (\mathbf{x}_0 - \mathbf{x}_c) - w_0 \hat{\mathbf{w}}$$

(where  $\mathbf{x}_c$  is the centroid of  $\mathcal{P}$ ). It is convenient to choose the origin of the  $(u, v)$  plane to be the point  $\boldsymbol{\rho}_0$ . The  $(u, v, w)$  coordinates of the  $i$ th panel vertex are then  $\bar{\mathbf{V}}_i = (u_i, v_i, 0)$  where

$$u_i = (\mathbf{V}_i - \mathbf{x}_c) \cdot \hat{\mathbf{u}}, \quad v_i = (\mathbf{V}_i - \mathbf{x}_c) \cdot \hat{\mathbf{v}}.$$

I also define a two-dimensional unit vector  $\hat{\boldsymbol{\ell}}_i$  pointing in the direction of the  $i$ th panel edge according to

$$\hat{\boldsymbol{\ell}}_i = \frac{\bar{\mathbf{V}}_{i+1} - \bar{\mathbf{V}}_i}{|\bar{\mathbf{V}}_{i+1} - \bar{\mathbf{V}}_i|}$$

and a two-dimensional unit vector  $\hat{\mathbf{m}}_i$  normal to the  $i$ th panel edge according to

$$\hat{\mathbf{m}}_i = \hat{\mathbf{w}} \times \hat{\boldsymbol{\ell}}_i.$$

## 2.2 Reduction of surface integrals to line integrals

I now convert the two-dimensional (surface) integrals  $\{\mathcal{I}, \mathcal{J}\}^p$  defined by (5b) into one-dimensional (line) integrals by using Stokes' theorem in the forms

$$\int_{\mathcal{P}} \nabla \cdot \mathbf{f}(\boldsymbol{\rho}) dA = \oint_{\partial\mathcal{P}} \mathbf{f}(\boldsymbol{\rho}) \cdot \hat{\mathbf{m}} dl \quad (7a)$$

$$\int_{\mathcal{P}} \nabla f(\boldsymbol{\rho}) dA = \oint_{\partial\mathcal{P}} f(\boldsymbol{\rho}) \hat{\mathbf{m}} dl. \quad (7b)$$

[Here  $\boldsymbol{\rho} = (u, v)$ .]

First consider the vector-valued function

$$\mathbf{f}(\boldsymbol{\rho}) = \frac{1}{p+2} \frac{[\rho^2 + w^2]^{(p+2)/2}}{\rho} \hat{\boldsymbol{\rho}}$$

with divergence

$$\nabla \cdot \mathbf{f}(\boldsymbol{\rho}) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho f_{\rho}(\boldsymbol{\rho})) = [\rho^2 + w^2]^{p/2}.$$

Applying (7a) to this function yields

$$\mathcal{I}^p \equiv \int_{\mathcal{P}} [\rho^2 + w^2]^{p/2} dA = \frac{1}{(p+2)} \int_{\partial\mathcal{P}} \frac{[\rho^2 + w^2]^{(p+2)/2}}{\rho} \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{m}} dl.$$

Next consider the scalar function

$$f(\rho) = \frac{1}{p+2} [\rho^2 + w^2]^{(p+2)/2}$$

with gradient

$$\nabla f(\rho) = \boldsymbol{\rho} [\rho^2 + w^2]^{p/2}.$$

Applying (7b) to this function yields

$$\mathcal{J}^p \equiv \int_{\mathcal{P}} \boldsymbol{\rho} \cdot r^p dA = \frac{1}{(p+2)} \int_{\partial\mathcal{P}} [\rho^2 + w^2]^{(p+2)/2} \hat{\mathbf{m}} dl.$$

## 2.3 Evaluation of line integrals

Line integrals are sums of integrals over line segments (edges). On edge  $i$ , we have

$$\boldsymbol{\rho}(s, t_i) = s\hat{\boldsymbol{\ell}}_i - t_i\hat{\mathbf{m}}_i, \quad \rho(s, t_i) = \sqrt{s^2 + t_i^2}, \quad \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{m}} = -\frac{t_i}{\rho}.$$

Here  $t_i$  is the perpendicular distance from edge  $\bar{\mathbf{V}}_i \bar{\mathbf{V}}_{i+1}$  to the origin and  $s$  runs from  $s_i^-$  to  $s_i^+$ , where

$$\begin{aligned} t_i &= -\mathbf{V}_i \cdot \hat{\mathbf{m}}_i = -\mathbf{V}_{i+1} \cdot \hat{\mathbf{m}}_i \\ s_i^- &= \mathbf{V}_i \cdot \hat{\boldsymbol{\ell}}_i \\ s_i^+ &= \mathbf{V}_{i+1} \cdot \hat{\boldsymbol{\ell}}_i \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{I}^p &= \frac{1}{(p+2)} \int_{\partial\mathcal{P}} \frac{[\rho^2 + w^2]^{(p+2)/2}}{\rho} \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{m}} \, dl \\ &= -\frac{1}{(p+2)} \sum_i t_i \underbrace{\int_{s_i^-}^{s_i^+} \frac{(s^2 + t_i^2 + w^2)^{(p+2)/2}}{(s^2 + t_i^2)} ds}_{I^p(s_i^-, s_i^+, t_i, w)} \\ \mathcal{J}^p &= \frac{1}{(p+2)} \int_{\partial\mathcal{P}} [\rho^2 + w^2]^{(p+2)/2} \hat{\mathbf{m}} \, d\hat{\boldsymbol{\ell}} \\ &= \frac{1}{(p+2)} \sum_i \hat{\mathbf{m}}_i \underbrace{\int_{s_i^-}^{s_i^+} [s^2 + t_i^2 + w^2]^{(p+2)/2} ds}_{J^p(s_i^-, s_i^+, t_i, w)}. \end{aligned}$$

The one-dimensional integrals defining  $I^p$  and  $J^p$  may be evaluated in closed form:

$p$	$I^p(s^-, s^+, t, w)$	$J^p(s^-, s^+, t, w)$
-5	$\frac{t}{w^2 X^2} \left( \frac{s^-}{R^-} - \frac{s^+}{R^+} \right) + \frac{\zeta}{w^3}$	$\frac{1}{X^2} \left( \frac{s^+}{R^+} - \frac{s^-}{R^-} \right)$
-3	$\frac{\zeta}{w}$	$\Lambda$
-1	$w\zeta + t\Lambda$	$\frac{1}{2} (R^+ s^+ - R^- s^- + X^2 \Lambda)$
+1	$\frac{t}{2} [R^+ s^+ - R^- s^- + \Lambda(t^2 + 3w^2)] + w^3 \zeta$	$\frac{1}{8} [2R^+ s^+ - 2R^- s^- + 5X^2(R^+ s^+ - R^- s^-) + 3\Lambda X^4]$

In this table, we have used the following shorthand:

$$\begin{aligned} X &= \sqrt{t^2 + w^2}, & R^+ &= \sqrt{s^{+2} + X^2}, & R^- &= \sqrt{s^{-2} + X^2} \\ Z^+ &= R^+ + s^+ & Z^- &= R^- + s^- \\ \Lambda &= \log \frac{Z^+}{Z^-}, & \zeta &= \text{atan} \left( \frac{ws^+}{tR^+} \right) - \text{atan} \left( \frac{ws^-}{tR^-} \right) \end{aligned}$$

## 2.4 Derivatives of reduced potentials

### 2.4.1 Potential derivatives from $\mathcal{I}, \mathcal{J}$ derivatives

When computing derivatives it is easiest to work first in the  $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$  coordinate system and later rotate back to the original coordinate system. In this case,

in-plane derivatives (derivatives with respect to the  $u, v$  coordinates) are distinguished from normal derivatives (derivatives with respect to the  $w$  coordinate). To highlight this distinction, in what follows the subscripts  $\alpha, \beta, \gamma$  will refer only to the in-plane coordinates ( $u, v$  coordinates), so that e.g.  $\partial_\alpha \mathcal{I}$  refers to an in-plane derivative, while  $\partial_w \mathcal{I}$  is a normal derivative. Note that the vector potential  $\mathbf{a}$ , like the quantities  $\mathcal{J}$  and  $\mathbf{Q}$ , has only in-plane components.

The starting point is equation (6):

$$p^{(n)} = \mathcal{I}^n, \quad a_\gamma^{(n)} = \mathcal{J}_\gamma^n - \overline{\mathbf{Q}}_\gamma \mathcal{I}^n$$

Derivatives of  $p$  are just derivatives of  $\mathcal{I}$ , computed as discussed below.

First derivatives of  $\mathbf{a}$  take the form

$$\begin{aligned} \partial_\beta a_\gamma^{(n)} &= \partial_\beta \mathcal{J}_\gamma^n + n \overline{\mathbf{Q}}_\gamma \mathcal{J}_\beta^{n-2} + \delta_{\beta\gamma} \mathcal{I}^n \\ \partial_w a_\gamma^{(n)} &= \partial_w \mathcal{J}_\gamma^n - \overline{\mathbf{Q}}_\gamma \partial_w \mathcal{I}^n \\ &= n w [\mathcal{J}_\gamma^{n-2} - \overline{\mathbf{Q}}_\gamma \mathcal{I}^{n-2}] \end{aligned}$$

When computing second derivatives of  $\mathbf{a}$ , it turns out that the double normal derivative  $\partial_w^2 a_\gamma$  and the mixed second partial  $\partial_w \partial_\alpha a_\gamma$  are straightforward to compute in terms of  $\mathcal{I}, \mathcal{J}$  and their first derivatives:

$$\begin{aligned} \partial_w \partial_\beta a_\gamma^{(n)} &= (n-2) w [\partial_\beta \mathcal{J}_\gamma^{(n-2)} + n \overline{\mathbf{Q}}_\gamma \mathcal{J}_\beta^{n-4}] + n w \delta_{\beta\gamma} \mathcal{I}^{n-2} \\ \partial_w^2 a_\gamma^{(n)} &= n [\mathcal{J}_\gamma^{n-2} - \overline{\mathbf{Q}}_\gamma \mathcal{I}^{n-2}] \\ &\quad + n(n-2) w^2 [\mathcal{J}_\gamma^{n-4} - \overline{\mathbf{Q}}_\gamma \mathcal{I}^{n-4}] \end{aligned}$$

The double in-plane derivative  $\partial_\alpha \partial_\beta a_\gamma$  is more difficult to compute. However, it turns out we don't need to compute this quantity as long as we are only interested in first derivatives of the  $\mathbf{E}$  and  $\mathbf{H}$  fields. To see this, note that second derivatives of  $\mathbf{a}$  only enter in the computation of first derivatives of  $\mathbf{H}$ , which involves differentiating the curl of  $\mathbf{a}$ . In the  $(uvw)$  system, the curl of  $\mathbf{a}$  reads

$$\nabla \times \mathbf{a} = -\partial_w a_v \hat{\mathbf{u}} + \partial_w a_u \hat{\mathbf{v}} + (\partial_u a_v - \partial_v a_u) \hat{\mathbf{w}}$$

The  $w$  derivative of this is

$$\partial_w (\nabla \times \mathbf{a}) = -\partial_w^2 a_v \hat{\mathbf{u}} + \partial_w^2 a_u \hat{\mathbf{v}} + (\partial_w \partial_u a_v - \partial_w \partial_v a_u) \hat{\mathbf{w}}$$

which does not require the double in-plane derivative. The in-plane derivative of  $\nabla \times \mathbf{a}$  is

$$\partial_\alpha (\nabla \times \mathbf{a}) = -\partial_\alpha \partial_w a_v \hat{\mathbf{u}} + \partial_\alpha \partial_w a_u \hat{\mathbf{v}} + (\partial_\alpha \partial_u a_v - \partial_\alpha \partial_v a_u) \hat{\mathbf{w}}$$



To elucidate the structure of the  $w$  component of this, I write

$$\begin{aligned}
\partial_u a_v^{(n)} - \partial_v a_u^{(n)} &= -n \int \underbrace{\left[ (\bar{\mathbf{x}})_u (\bar{\mathbf{x}} - \bar{\mathbf{Q}})_v - (\bar{\mathbf{x}})_v (\bar{\mathbf{x}} - \bar{\mathbf{Q}})_u \right]}_{(\bar{\mathbf{Q}})_u (\bar{\mathbf{x}})_v - (\bar{\mathbf{Q}})_v (\bar{\mathbf{x}})_u} r^{n-2} dA \\
&= -n \left[ (\bar{\mathbf{Q}})_u \int (\bar{\mathbf{x}})_v r^{n-2} dA - (\bar{\mathbf{Q}})_v \int (\bar{\mathbf{x}})_u r^{n-2} dA \right] \\
&= -n \left[ (\bar{\mathbf{Q}})_u \mathcal{J}_v^{n-2} - (\bar{\mathbf{Q}})_v \mathcal{J}_u^{n-2} \right]
\end{aligned}$$

and thus

$$\partial_\alpha (\nabla \times \mathbf{a})_w = -n \left[ (\bar{\mathbf{Q}})_u \partial_\alpha \mathcal{J}_v^{n-2} - (\bar{\mathbf{Q}})_v \partial_\alpha \mathcal{J}_u^{n-2} \right].$$

The upshot is that all quantities needed to compute first and second derivatives of the potentials may be obtained from the  $\mathcal{I}, \mathcal{J}$  integrals and their first derivatives.

#### 2.4.2 Derivatives of $\mathcal{I}, \mathcal{J}$ integrals

(In what follows, subscripts  $\mu, \nu$  refer to derivatives with respect to coordinates in the plane of the panel [ $u, v$  derivatives in the  $(u, v, w)$  system], as distinct from  $w$  derivatives, which are directional derivatives in the direction normal to the panel.)

Derivatives of the  $\mathcal{I}$  integrals, and the normal derivative of the  $\mathcal{J}$  integrals, may be carried out at the level of surface integrals:

$$\begin{aligned}
\partial_\mu \mathcal{I}^p(\mathbf{x}) &= \partial_\mu \int_{\mathcal{P}} [\rho^2 + w^2]^{p/2} dA = -p \int_{\mathcal{P}} \rho_\mu [\rho^2 + w^2]^{(p-2)/2} dA \\
&= -p \mathcal{J}_\mu^{p-2}(\mathbf{x}) \\
\partial_w \mathcal{I}^p(\mathbf{x}) &= \partial_w \int_{\mathcal{P}} [\rho^2 + w^2]^{p/2} dA = pw \partial_w \int_{\mathcal{P}} [\rho^2 + w^2]^{(p-2)/2} dA \\
&= pw \mathcal{I}^{p-2}(\mathbf{x})
\end{aligned}$$

and similarly

$$\partial_w \mathcal{J}^p(\mathbf{x}) = pw \mathcal{J}^{p-2}(\mathbf{x})$$

In-plane derivatives of the  $\mathcal{J}$  integrals are easiest to carry out at the level of

line integrals:

$$\begin{aligned}\partial_\mu \mathcal{J}_\nu^p(\mathbf{x}) &= \frac{1}{(p+2)} \sum_i \left\{ (\partial_\mu J^p) \hat{m}_{i\nu} \right\} \\ \partial_\mu J^p(s_i^-, s_i^+, t_i, w) &= \left[ \frac{\partial J^p}{\partial \hat{\ell}_i} \hat{\ell}_i + \frac{\partial J^p}{\partial \hat{\mathbf{m}}_i} \hat{\mathbf{m}}_i \right] \\ \frac{\partial J^p}{\partial \hat{\ell}_i} &= -(s_i^{-2} + t_i^2 + w^2)^{(p+2)/2} - (s_i^{+2} + t_i^2 + w^2)^{(p+2)/2} \\ \frac{\partial J^p}{\partial \hat{\mathbf{m}}_i} &= (p+2)t_i J^{p-2}\end{aligned}$$

### 2.4.3 Derivatives of desingularized terms

We have

$$G^{\text{DS}}(r) = \frac{\text{ExpRelBar}(ikr, 3)}{4\pi r}$$

and thus

$$\begin{aligned}\partial_i G^{\text{DS}}(r) &= r_i \left[ ik \frac{\text{ExpRelBar}(ikr, 2)}{4\pi r^2} - \frac{\text{ExpRelBar}(ikr, 3)}{4\pi r^3} \right] \\ \partial_i \partial_j G^{\text{DS}}(r) &= \delta_{ij} \left[ ik \frac{\text{ExpRelBar}(ikr, 2)}{4\pi r^2} - \frac{\text{ExpRelBar}(ikr, 3)}{4\pi r^3} \right] \\ &\quad + r_i r_j \left[ (ik)^2 \frac{\text{ExpRelBar}(ikr, 1)}{4\pi r^3} - 3ik \frac{\text{ExpRelBar}(ikr, 2)}{4\pi r^4} + 3 \frac{\text{ExpRelBar}(ikr, 3)}{4\pi r^5} \right]\end{aligned}$$

### 3 Far fields at nearby points

The contribution of a single panel  $\mathcal{P}$  to the reduced fields may be written in an alternative way using the dyadic Green's functions  $\mathbf{G}(\mathbf{r})$ ,  $\mathbf{C}(\mathbf{r})$

$$e_i(\mathbf{x}) = \int G_{ij}(\mathbf{x}, \mathbf{x}') b_j(\mathbf{x}') d\mathbf{x}', \quad h_i(\mathbf{x}) = -ik \int C_{ij}(\mathbf{x}, \mathbf{x}') b_j(\mathbf{x}') d\mathbf{x}'$$

Retaining only far-field contributions,

$$G_{ij} = \left( \delta_{ij} + \frac{r_i r_j}{r^2} \right) \frac{e^{ikr}}{4\pi r}, \quad -ikC_{ij} = -ik\varepsilon_{ijk} \frac{r_k}{r} \frac{e^{ikr}}{4\pi r}$$

Separate  $e_i$  into singular and non-singular contributions:

$$\begin{aligned} \mathbf{e}(\mathbf{x}) &= \frac{\ell}{8\pi A} \left[ \mathbf{e}^{(-1)}(\mathbf{x}) + \mathbf{e}^{\text{PS}}(\mathbf{x}) \right] \\ e_i^{(-1)}(\mathbf{x}) &= \int \left( \frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3} \right) (\mathbf{x} - \mathbf{Q})_j dr \\ e_i^{\text{PS}}(\mathbf{x}) &= \int \left( \delta_{ij} + \frac{r_i r_j}{r^2} \right) \frac{\text{ExpRelBar}(ikr, 1)}{r} (\mathbf{x} - \mathbf{Q})_j dr \end{aligned}$$

The contributions to  $\mathbf{e}^{(-1)}$  are easiest to work out in the coordinate system of  $\mathcal{P}$ . The first term is

$$\begin{aligned} e_\mu^{(-1)a}(\mathbf{x}) &= \int \frac{(\mathbf{x}' - \mathbf{Q})_\mu}{r} dA \\ &= a_\mu^{(-1)} \end{aligned}$$

The second term is

$$\begin{aligned} e_\mu^{(-1)b}(\mathbf{x}) &= \int \frac{(\mathbf{x}' - \mathbf{x})_\mu (\mathbf{x}' - \mathbf{x})_\nu (\mathbf{x}' - \mathbf{Q})_\nu}{r^3} dA \\ &= \int \frac{(\mathbf{x}' - \mathbf{x})_\mu}{r} dA - (\overline{\mathbf{Q}})_\nu \int \frac{(\mathbf{x}' - \mathbf{x})_\mu (\mathbf{x}' - \mathbf{x})_\nu}{r^3} dA \end{aligned}$$

The  $w$  component of this is

$$e_w^{(-1)b}(\mathbf{x}) = w\mathcal{L}^{-1} - w(\overline{\mathbf{Q}})_\nu \mathcal{J}_\nu^{-3}$$