Electrostatics of a Spherical Dielectric Shell

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Consider a spherical dielectric shell, with relative dielectric constant ϵ and exterior and interior radii $R_{\rm E}$ and $R_{\rm I}$, placed in a constant electric field pointing in positive z direction,

$$\mathbf{E}^{\rm ext} = E_0\hat{\mathbf{z}}.$$

How does the presence of the shell modify the electric fields?

Potential *ansatz*

$$\phi(r,\theta) = \begin{cases} -E_0 r\cos\theta + \frac{A\cos\theta}{r^2}, & r \geq R_{\rm E} \\ Br\cos\theta + \frac{C\cos\theta}{r^2}, & R_{\rm I} \leq \; r <\leq R_{\rm E} \\ Dr\cos\theta, & r \leq R_{\rm I} \end{cases} \tag{1}$$

Boundary conditions

$$\phi\big(R_{\rm E}^+\big) = \phi\big(R_{\rm E}^-\big) \tag{2a}$$

$$\left|\frac{\partial\phi}{\partial r}\right|_{R_{\rm E}^+} = \epsilon\left|\frac{\partial\phi}{\partial r}\right|_{R_{\rm E}^-} \tag{2b}$$

$$\phi\big(R_{\rm I}^+\big) = \phi\big(R_{\rm I}^-\big) \tag{2c}$$

$$\epsilon\left|\frac{\partial\phi}{\partial r}\right|_{R_{\rm I}^+} = \left|\frac{\partial\phi}{\partial r}\right|_{R_{\rm I}^-} \tag{2d}$$

Linear System

To simplify the remaining discussion, I take $R_{\rm E} \equiv 1$ and put $R_{\rm I}/R_{\rm E} \equiv \gamma$. Inserting (1) into (2) yields a 4×4 linear system:

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -2 & -\epsilon & 2\epsilon & 0 \\ 0 & \gamma & \gamma^{-2} & -\gamma \\ 0 & \epsilon & -2\epsilon\gamma^{-3} & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} E_0 \\ E_0 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

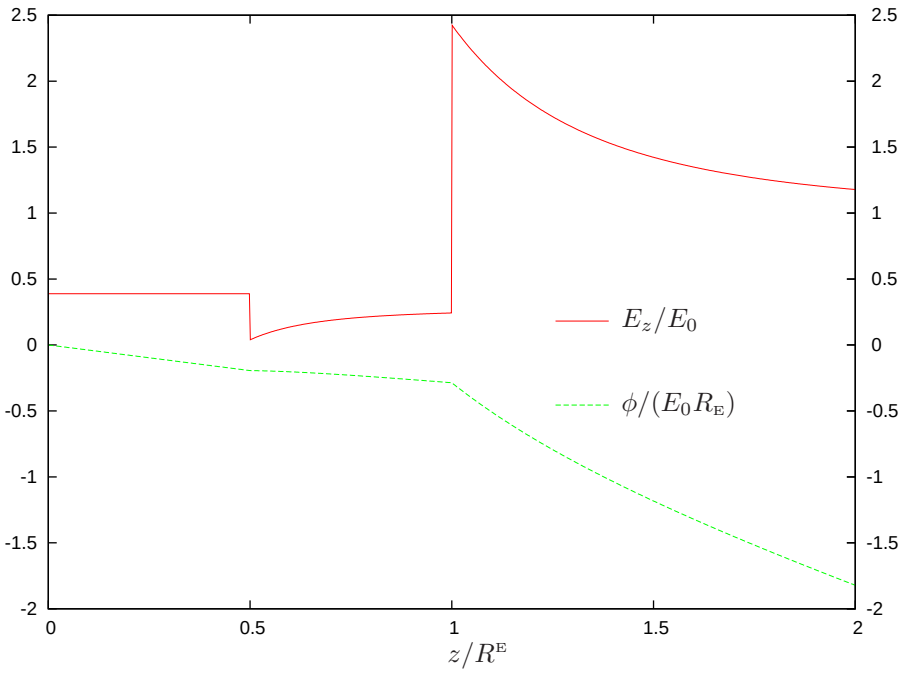

Figure 1: Potential and z component of electric field along the z axis for the case $\epsilon = 10$.

Solution

For the particular case $\gamma = 0.5$, the solution to (3) is

$$A = \frac{7(\epsilon-1)(1+2\epsilon)}{2(7\epsilon^2+22\epsilon+7)}E_0$$
$$B = -\frac{12(1+2\epsilon)}{7\epsilon^2+22\epsilon+7}E_0$$
$$C = -\frac{3(\epsilon-1)}{2(7\epsilon^2+22\epsilon+7)}E_0$$
$$D = -\frac{36\epsilon}{7\epsilon^2+22\epsilon+7}E_0$$

The potential and z-component of the electric field along the z-axis are plotted in Figure 1 for the case $\epsilon = 10$.