Implicit handling of multilayered material substrates in full-wave SCUFF-EM calculations

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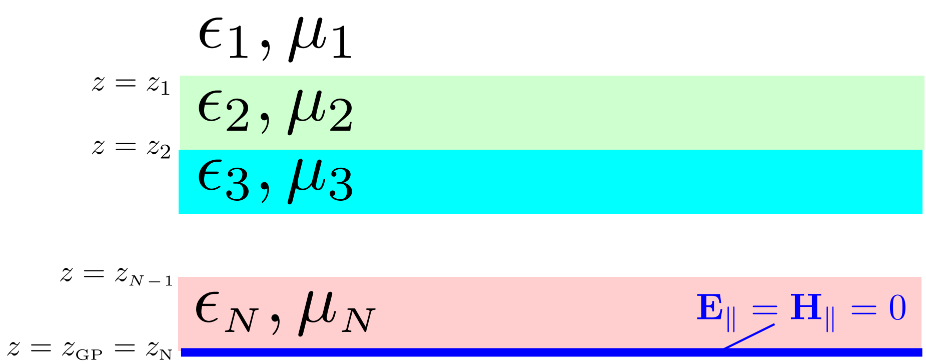

Figure 1: Geometry of the layered substrate. The nth layer has relative permittivity and permeability ϵ_n, μ_n, and its lower surface lies at $z = z_n$. The ground plane, if present, lies at $z = z_{\rm GP}$.

1 Overview

In a previous memo[1] I considered SCUFF-STATIC electrostatics calculations in the presence of a multilayered dielectric substrate. In this memo I extend that discussion to the case of *full-wave* (i.e. nonzero frequencies beyond the quasistatic regime) scattering calculations in the SCUFF-EM core library.

Substrate geometry

As shown in Figure 1, I consider a multilayered substrate consisting of N material layers possibly terminated by a perfectly-conducting ground plane. The uppermost layer (layer 1) is the infinite half-space above the substrate. The nth layer has relative permittivity and permeability ϵ_n, μ_n, and its lower surface lies at $z = z_n$. The ground plane, if present, lies at $z \equiv z_{\rm N} \equiv z_{\rm GP}$. If the ground plane is absent, layer N is an infinite half-space.[2]

Definition of the substrate DGF

I will use the symbol $\mathbf{\Gamma}(\omega; \mathbf{x}_{\rm D}, \mathbf{x}_{\rm S})$ for the *total* 6×6 dyadic Green's function relating time-harmonic fields at $\mathbf{x}_{\rm D}$ to sources at $\mathbf{x}_{\rm S}$: thus, if $\boldsymbol{\mathcal{S}} \equiv \binom{\mathbf{J}}{\mathbf{M}}$ is the 6-vector distribution of free electric and magnetic currents in the presence of the substrate, then the 6-vector of electric and magnetic fields $\boldsymbol{\mathcal{F}} \equiv \binom{\mathbf{E}}{\mathbf{H}}$ is given by

$$\boldsymbol{\mathcal{F}}(\mathbf{x}_{\rm D}) = \int \mathbf{\Gamma}(\mathbf{x}_{\rm D}, \mathbf{x}_{\rm S}) \cdot \boldsymbol{\mathcal{S}}(\mathbf{x}_{\rm S}) d\mathbf{x}_{\rm S}.$$

[1] "Implicit handling of multilayered dielectric substrates in SCUFF-STATIC"

[2] As in the electrostatic case, this means that a finite-thickness substrate consisting of N material layers is described as a stack of $N+1$ layers in which the bottommost layer is an infinite half-space ($z_{\rm N+1} = -\infty$) with the material properties of vacuum ($\epsilon_{N+1} = \mu_{N+1} = 1$).

The 6×6 tensor $\mathbf{\Gamma}$ has a 2×2 block structure:

$$\mathbf{\Gamma}=\left(\begin{array}{cc}\mathbf{\Gamma}^{\mathrm{EE}} & \mathbf{\Gamma}^{\mathrm{EM}}\\ \mathbf{\Gamma}^{\mathrm{ME}} & \mathbf{\Gamma}^{\mathrm{MM}}\end{array}\right) \tag{1a}$$

with the 3×3 subblocks defined by

$$\Gamma^{\mathrm{PQ}}_{ij}(\omega,\mathbf{x}_{\mathrm{D}},\mathbf{x}_{\mathrm{S}})=\left(\begin{array}{l}i\text{-component of P-type field at }\mathbf{x}_{\mathrm{D}}\text{ due to }j\text{-directed}\\ \text{Q-type point current source at }\mathbf{x}_{\mathrm{S}}\text{, all fields and}\\ \text{sources having time dependence }\sim e^{-i\omega t}\end{array}\right) \tag{1b}$$

Homogeneous DGF In an infinite *homogeneous* medium with relative permittivity and permeability $\{\epsilon^r,\mu^r\}$, $\mathbf{\Gamma}$ reduces to its homogeneous form, for which I will use the symbol $\mathbf{\Gamma}^{0r}$ (where the r index labels the medium, which in this case will be one of the layers in Figure 1, i.e. $r\in\{1,2,\cdots,N\}$):

$$\mathbf{x}_{\mathrm{D}},\mathbf{x}_{\mathrm{S}}\in\text{infinite homogeneous medium }r\quad\Longrightarrow\quad\mathbf{\Gamma}(\omega;\mathbf{x}_{\mathrm{D}},\mathbf{x}_{\mathrm{S}})=\mathbf{\Gamma}^{0r}(\omega;\mathbf{x}_{\mathrm{D}}-\mathbf{x}_{\mathrm{S}})$$

where[3]

$$\mathbf{\Gamma}^{0r}(\omega,\mathbf{r})\equiv\left(\begin{array}{cc}ik_rZ_0Z^r\mathbf{G}(k_r,\mathbf{r}) & ik_r\mathbf{C}(k_r,\mathbf{r})\\ -ik_r\mathbf{C}(k_r,\mathbf{r}) & \frac{ik_r}{Z_0Z^r}\mathbf{G}(k_r,\mathbf{r})\end{array}\right) \tag{2}$$

$$k_r\equiv\sqrt{\epsilon_0\epsilon^r\mu_0\mu^r}\cdot\omega,\qquad Z_0Z^r\equiv\sqrt{\frac{\mu_0\mu^r}{\epsilon_0\epsilon^r}},$$

$$G_{ij}=\left(\delta_{ij}-\frac{1}{k^2}\partial_i\partial_j\right)\frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|},\qquad C_{ij}=\frac{\varepsilon_{i\ell m}}{ik}\partial_\ell G_{mj}$$

Inhomogeneous DGF On the other hand, in the presence of the multilayered substrate the full DGF $\mathbf{\Gamma}$ receives corrections, which may be thought of as the fields radiated by surface currents induced on the interfacial surfaces of the substrate, and which I will denote by the symbol $\boldsymbol{\mathcal{G}}$:

$$\mathbf{\Gamma}(\mathbf{x}_{\mathrm{D}},\mathbf{x}_{\mathrm{S}})=\boldsymbol{\mathcal{G}}(\mathbf{x}_{\mathrm{D}},\mathbf{x}_{\mathrm{S}})+\begin{cases}\mathbf{\Gamma}^{0r}(\mathbf{x}_{\mathrm{D}}-\mathbf{x}_{\mathrm{S}}), & \mathbf{x}_{\mathrm{S}}\in\text{layer r}\\ 0, & \text{otherwise}\end{cases} \tag{3}$$

Like $\mathbf{\Gamma}$, $\boldsymbol{\mathcal{G}}$ is a 6×6 matrix with a 2×2 block structure:

$$\boldsymbol{\mathcal{G}}(\omega;\mathbf{x}_{\mathrm{D}},\mathbf{x}_{\mathrm{S}})=\left(\begin{array}{cc}\boldsymbol{\mathcal{G}}^{\mathrm{EE}} & \boldsymbol{\mathcal{G}}^{\mathrm{EM}}\\ \boldsymbol{\mathcal{G}}^{\mathrm{ME}} & \boldsymbol{\mathcal{G}}^{\mathrm{MM}}\end{array}\right) \tag{4}$$

with the 3×3 subblocks defined by

$$\mathcal{G}^{\mathrm{PQ}}_{ij}=\left(\begin{array}{l}i\text{-component of P-type field at }\mathbf{x}_{\mathrm{D}}\text{ due to surface currents on sub-}\\ \text{strate interface layers induced by }j\text{-directed Q-type source at }\mathbf{x}_{\mathrm{S}}.\end{array}\right)$$

LIBSUBSTRATE is a code for numerical computation of $\boldsymbol{\mathcal{G}}$.

[3] Cf. Section 3 of the companion memo "LIBSCUFF implementation and Technical Details," `http://homerreid.github.io/scuff-em-documentation/tex/lsInnards.pdf`

Organization of SCUFF-EM implementation and this memo

The full-wave substrate implementation in SCUFF-EM consists of multiple working parts that fit together in a somewhat modular fashion.

Roughly speaking, the computational problem may be divided into two parts:

(a) For given source and evaluation (or "destination") points $\{\mathbf{x}_{\text{S}}, \mathbf{x}_{\text{D}}\}$ at a given angular frequency ω in the presence of a multilayer substrate, numerically compute the substrate DGF correction $\boldsymbol{\mathcal{G}}(\omega, \mathbf{x}_{\text{D}}, \mathbf{x}_{\text{S}})$. This task is independent of SCUFF-EM and is implemented by a standalone library called LIBSUBSTRATE, described in Section 2 of this memo.

(b) For a SCUFF-EM geometry in the presence of a substrate, compute the substrate corrections to the BEM system matrix $\mathbf{M}$ and RHS vector $\mathbf{v}$, as well as the substrate corrections to post-processing quantities such as scattered fields. This is done by the file `Substrate.cc` in LIBSCUFF and is described in Section 3 of this memo.

2 LIBSUBSTRATE: Numerical computation of substrate Green's functions

Numerical evaluation of substrate contributions to dyadic Green's functions is handled by a C++ library called LIBSUBSTRATE. Although this library is packaged and distributed with SCUFF-EM and depends on other support libraries in the SCUFF-EM distribution, it is independent of the particular integral-equation formulation implemented by LIBSCUFF, and thus should be of general utility beyond SCUFF-EM.

2.1 Overview of computational strategy

LIBSUBSTRATE decomposes the problem of computing $\boldsymbol{\mathcal{G}}$ into several logical steps, as follows:

1. Solve a linear system to obtain the Fourier-space representation $\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q})$. Here $\mathbf{q} = (q_x, q_y)$ is a 2D Fourier variable. (Section 2.2.)

2. Reduce the two-dimensional integral over $\mathbf{q}$ to a one-dimensional integral over $|\mathbf{q}| \equiv q$. (Section 2.3.)

3. Evaluate the q integral using established methods for evaluating Sommerfeld integrals. (Section **??**.)

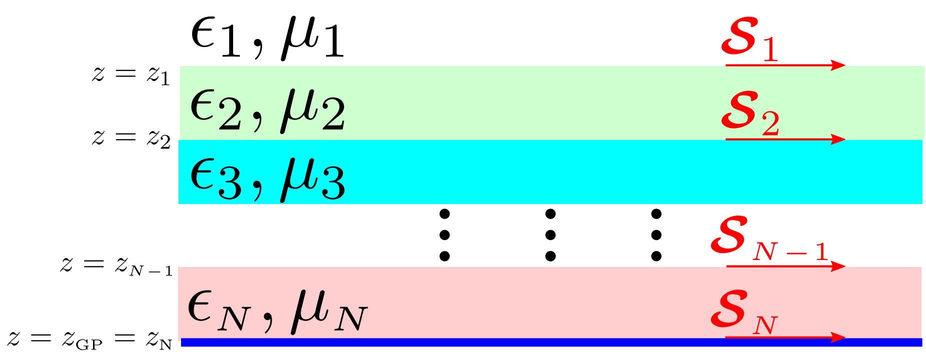

Figure 2: Effective surface-current approach to treatment of multilayer substrate. External field sources induce a distribution of electric and magnetic surface currents $\boldsymbol{\mathcal{S}}_n = \binom{\mathbf{K}_n}{\mathbf{N}_n}$ on the nth material interface, and the fields radiated by these effective currents account for the disturbance presented by the substrate.

2.2 Computation of Fourier-space DGF $\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q})$

To compute the substrate correction to the fields of external sources, I consider the effective tangential electric and magnetic surface currents $\mathbf{K}$ and $\mathbf{N}$ induced on the interfacial layers by the external field sources (Figure 2). This is the direct extension to full-wave problems of the formalism I used in the electrostatic case, and it comports well with the spirit of surface-integral-equation methods.

More specifically, on the material interface layer at $z = z_n$ I have a four-vector surface-current density $\boldsymbol{\mathcal{S}}_n(\boldsymbol{\rho})$, where $\boldsymbol{\rho} = (x, y)$ and the components of $\boldsymbol{\mathcal{S}}$ are

$$\boldsymbol{\mathcal{S}}_n(\boldsymbol{\rho}) = \left(\begin{array}{c} K_x(\boldsymbol{\rho}) \\ K_y(\boldsymbol{\rho}) \\ N_x(\boldsymbol{\rho}) \\ N_y(\boldsymbol{\rho}) \end{array}\right). \tag{5}$$

Fields in layer interiors. I will adopt the convention that the lower (upper) bounding surface for each region is the positive (negative) bounding surface for that region in the usual sense of SCUFF-EM regions and surfaces (in which the sign of a {surface,region} pair $\{\mathcal{S}, \mathcal{R}\}$ is the sign with which surface currents on $\mathcal{S}$ contribute to fields in $\mathcal{R}$). Thus, at a point $\mathbf{x} = (\boldsymbol{\rho}, z)$ in the interior of layer n $(z_{n-1} > z > z_n)$, the six-vector of total fields $\boldsymbol{\mathcal{F}} = \left(\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right)$ reads

$$\boldsymbol{\mathcal{F}}_n(\boldsymbol{\rho}, z) = -\boldsymbol{\Gamma}^{0n}(z_{n-1}) \star \boldsymbol{\mathcal{S}}_{n-1} + \boldsymbol{\Gamma}^{0n}(z_n) \star \boldsymbol{\mathcal{S}}_n + \boldsymbol{\mathcal{F}}_n^{\rm ext}(\boldsymbol{\rho}, z) \tag{6}$$

where $\boldsymbol{\mathcal{F}}_n^{\rm ext}$ are the externally-sourced (incident) fields due to sources in layer n, $\boldsymbol{\Gamma}^{0n}$ is the 6×6 homogeneous dyadic Green's function for material layer n, and $\star$ is shorthand for the convolution operation

$$\text{“}\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},z)\equiv\boldsymbol{\Gamma}(z')\star\boldsymbol{\mathcal{S}}\text{''}\qquad\Longrightarrow\qquad\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},z)=\int\boldsymbol{\Gamma}(\boldsymbol{\rho}-\boldsymbol{\rho}',z-z')\cdot\boldsymbol{\mathcal{S}}(\boldsymbol{\rho}')d\boldsymbol{\rho}' \tag{7}$$

where the integral extends over the entire interfacial plane. I will evaluate convolutions of this form using the 2D Fourier representation of $\boldsymbol{\Gamma}^{0n}$:

$$\boldsymbol{\Gamma}^{0n}(\boldsymbol{\rho},z)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\widetilde{\boldsymbol{\Gamma}^{0n}}(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}} \tag{8a}$$

$$\widetilde{\boldsymbol{\Gamma}^{0n}}(\mathbf{q},z)=\frac{1}{2}\left(\begin{array}{cc}-\frac{\omega\mu_0\mu_n}{q_{zn}}\widetilde{\mathbf{G}}^{\pm} & +\widetilde{\mathbf{C}}^{\pm}\\ -\widetilde{\mathbf{C}}^{\pm} & -\frac{\omega\epsilon_0\epsilon_n}{q_{zn}}\widetilde{\mathbf{G}}^{\pm}\end{array}\right)e^{iq_z|z|} \tag{8b}$$

$$\widetilde{\mathbf{G}}^{\pm}(\mathbf{q},k)=\left(\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right)-\frac{1}{k^2}\left(\begin{array}{ccc}q_x^2 & q_xq_y & \pm q_xq_z\\ q_yq_x & q_y^2 & \pm q_yq_z\\ \pm q_zq_x & \pm q_zq_y & q_z^2\end{array}\right) \tag{8c}$$

$$\widetilde{\mathbf{C}}^{\pm}(\mathbf{q},k)=\left(\begin{array}{ccc}0 & \mp 1 & +q_y/q_z\\ \pm 1 & 0 & -q_x/q_z\\ -q_y/q_z & +q_x/q_z & 0\end{array}\right) \tag{8d}$$

$$k_n\equiv\sqrt{\epsilon_0\epsilon_n\mu_0\mu_n}\cdot\omega,\qquad q_z\equiv\sqrt{k^2-|\mathbf{q}|^2},\qquad \pm=\text{sign }z. \tag{8e}$$

With this representation, convolutions like (7) become products in Fourier space:

$$\boldsymbol{\Gamma}(z')\star\boldsymbol{\mathcal{S}}=\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},z)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}},\quad\text{with}\quad\widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{q},z)=\widetilde{\boldsymbol{\Gamma}}(\mathbf{q},z-z')\widetilde{\boldsymbol{\mathcal{S}}}(\mathbf{q})$$

Surface currents from incident fields. To determine the surface currents induced by given incident-field sources, I apply boundary conditions. The boundary condition at $z=z_n$ is that the tangential $\mathbf{E},\mathbf{H}$ fields be continuous: in Fourier space, we have

$$\widetilde{\boldsymbol{\mathcal{F}}}_{\parallel}(\mathbf{q},z=z_n^+)=\widetilde{\boldsymbol{\mathcal{F}}}_{\parallel}(\mathbf{q},z=z_n^-) \tag{9}$$

The fields just **above** the interface ($z\to z_n^+$) receive contributions from three sources:

- Surface currents at $z=z_{n-1}$, which contribute with a minus sign and via the Green's function for region n;
- Surface currents at $z=z_n$, which contribute with a plus sign and via the Green's function for region n; and
- external field sources in region n.

The fields just **below** the interface ($z = z_n^-$) receive contributions from three sources:

- Surface currents at $z = z_n$, which contribute with a minus sign and via the Green's function for region $n+1$;
- Surface currents at $z = z_{n+1}$, which contribute with a plus sign and via the Green's function for region $n+1$; and
- external field sources in region $n+1$.

Then equation (9) reads (temporarily omitting $\mathbf{q}$ arguments)

$$\begin{aligned}&-\widetilde{\boldsymbol{\Gamma}^{0n}}_{\parallel}(z_n - z_{n-1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n-1} + \widetilde{\boldsymbol{\Gamma}^{0n}}_{\parallel}(0^+)\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n} + \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n\parallel}(z_n)\\ &\qquad = -\widetilde{\boldsymbol{\Gamma}^{0,n+1}}\|(0^-)\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n} + \widetilde{\boldsymbol{\Gamma}^{0,n+1}}\|(z_n - z_{n+1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n+1} + \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n+1\parallel}(z_n)\end{aligned}$$

or

$$\mathbf{M}_{n,n-1}\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n-1} + \mathbf{M}_{n,n}\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n} + \mathbf{M}_{n,n+1}\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n+1} = \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n+1\parallel}(z_n) - \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n\parallel}(z_n) \qquad (10)$$

with the 4×4 matrix blocks[4]

$$\mathbf{M}_{n,n-1} = -\widetilde{\boldsymbol{\Gamma}^{0n}}_{\parallel}(z_n - z_{n-1}) \qquad (13a)$$

$$\mathbf{M}_{n,n} = +\widetilde{\boldsymbol{\Gamma}^{0n}}_{\parallel}(0^+) + \widetilde{\boldsymbol{\Gamma}^{0,n+1}}_{\parallel}(0^-) \qquad (13b)$$

$$\mathbf{M}_{n,n+1} = -\widetilde{\boldsymbol{\Gamma}^{0,n+1}}_{\parallel}(z_n - z_{n+1}) \qquad (13c)$$

Writing down equation (10) equation for all N dielectric interfaces yields a $4N\times 4N$ system of linear equations, with triadiagonal 4×4 block form, relating the surface currents on all layers to the external fields due to sources in all regions:

$$\mathbf{M}\cdot\mathbf{s} = \mathbf{f} \qquad (14)$$

[4]The 4×4 $\mathbf{M}$ blocks here have 2×2 block structure:

$$\mathbf{M}_{n,n} = \sum_{r\in\{n,n+1\}}\frac{1}{2}\begin{pmatrix} -\frac{\omega\epsilon_r}{Z_0 q_{zr}}\mathbf{g}(k_r,\mathbf{q}) & 0 \\ 0 & -\frac{\omega\mu_r Z_0}{q_{zr}}\mathbf{g}(k_r,\mathbf{q})\end{pmatrix} \qquad (11)$$

$$\mathbf{M}_{n,n\pm 1} = \frac{1}{2}\begin{pmatrix} -\frac{\omega\epsilon_r}{Z_0 q_{zr}}\mathbf{g}(k_r,\mathbf{q}) & \mathbf{c}^{\pm} \\ -\mathbf{c}^{\pm} & -\frac{\omega\mu_r Z_0}{q_{zn^*}}\mathbf{g}(k_r,\mathbf{q})\end{pmatrix} e^{iq_{zr}|z_n - z_{n\pm 1}|} \qquad (12)$$

where I put $r \equiv \begin{cases} n, & \text{for } \mathbf{M}_{n,n-1} \\ n+1, & \text{for } \mathbf{M}_{n,n+1}\end{cases}$. and

$$\mathbf{g}(k;\mathbf{q}) = \mathbf{1} - \frac{\mathbf{q}\mathbf{q}^{\top}}{k^2}, \qquad \mathbf{c}^{\pm} = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0\end{pmatrix}$$

where $\mathbf{M}$ is the $4N\times 4N$ block-tridiagonal matrix (13) and where the $4N$-vectors $\mathbf{s}$, $\mathbf{f}$ read

$$\mathbf{s}=\left(\begin{array}{c}\widetilde{\boldsymbol{\mathcal{S}}}_1\\ \widetilde{\boldsymbol{\mathcal{S}}}_2\\ \widetilde{\boldsymbol{\mathcal{S}}}_3\\ \vdots\\ \widetilde{\boldsymbol{\mathcal{S}}}_N\end{array}\right),\qquad \mathbf{f}=\left(\begin{array}{c}-\widetilde{\boldsymbol{\mathcal{F}}}_{1\parallel}(z_1)+\widetilde{\boldsymbol{\mathcal{F}}}_{2\parallel}(z_1)\\ -\widetilde{\boldsymbol{\mathcal{F}}}_{2\parallel}(z_2)+\widetilde{\boldsymbol{\mathcal{F}}}_{3\parallel}(z_2)\\ -\widetilde{\boldsymbol{\mathcal{F}}}_{3\parallel}(z_3)+\widetilde{\boldsymbol{\mathcal{F}}}_{3\parallel}(z_4)\\ \vdots\\ -\widetilde{\boldsymbol{\mathcal{F}}}_{N-1,\parallel}(z_{N-1})+\widetilde{\boldsymbol{\mathcal{F}}}_{N\parallel}(z_{N-1})\end{array}\right).$$

Solving (14) yields the induced surface currents on all layers in terms of the incident fields:

$$\mathbf{s}=\mathbf{W}\cdot\mathbf{f}\quad\text{where}\quad\mathbf{W}\equiv\mathbf{M}^{-1}$$

or, more explicitly,

$$\widetilde{\boldsymbol{\mathcal{S}}}_n=\sum_m W_{nm}\mathbf{f}_m \tag{15}$$

Surface currents induced by point sources

For DGF computations the incident fields arise from a single point source—say, a j-directed source in region s. Then the only nonzero length-4 blocks of the RHS vector in (14) are $\mathbf{f}_{s-1},\mathbf{f}_{\mathrm{S}}$ with components ($\ell=\{1,2,4,5\}$)

$$\Big(\mathbf{f}_{s-1}\Big)_\ell=-\widetilde{\Gamma}^{0s}_{\ell j}(z_{s-1}-z_{\mathrm{S}}),\qquad \Big(\mathbf{f}_{s}\Big)_\ell=+\widetilde{\Gamma}^{0s}_{\ell j}(z_{\mathrm{S}}-z_{\mathrm{S}}) \tag{16}$$

and the surface currents on interface layer n are obtained by solving (15):

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{S}}}_n&=\mathbf{W}_{n,s-1}\,\mathbf{f}_{s-1}+\mathbf{W}_{n,s}\,\mathbf{f}_s\\ &=\sum_{p=0}^{1}(-1)^{p+1}\mathbf{W}_{n,s-1+p}\cdot\widetilde{\mathbf{\Gamma}^{0s}}_{\parallel,j}(z_{\mathrm{S}}-z_{s-1+p})\end{aligned} \tag{17}$$

Fields due to surface currents

Given the surface currents induced by a j-directed point source at $\mathbf{x}_{\mathrm{S}}$, I evaluate the fields due to these currents to get the substrate DGF contribution $\boldsymbol{\mathcal{G}}$. If the evaluation point $\mathbf{x}_{\mathrm{D}}$ lies in region d, then the fields receive contributions from the surface currents at z_{d-1} and z_{D}, propagated by the homogeneous DGF for region d:

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{F}}}(z_{\mathrm{D}})&=-\widetilde{\mathbf{\Gamma}^{0d}}(z_{\mathrm{D}}-z_{d-1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{d-1}+\widetilde{\mathbf{\Gamma}^{0d}}(z_{\mathrm{D}}-z_{\mathrm{D}})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_d\\ &=\sum_{q=0}^{1}(-1)^{q+1}\widetilde{\mathbf{\Gamma}^{0d}}(z_{\mathrm{D}}-z_{d+q-1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{d+q-1}\end{aligned} \tag{18}$$

(The minus sign in the first term arises because, in my convention, surface currents on the upper surface of a region contribute to the fields in that region with a minus sign). Inserting (17), the i component here—which is the ij component of the substrate DGF—is

$$\widetilde{\mathcal{G}}_{ij}(z_{\text{D}}, z_{\text{S}}) = \sum_{p,q=0}^{1} (-1)^{p+q} \widetilde{\mathbf{\Gamma}^{0d}}_{i,\parallel}(z_{\text{D}} - z_{d-1+q}) \mathbf{W}_{d-1+q,s-1+p} \widetilde{\mathbf{\Gamma}^{0s}}_{\parallel,j}(z_{s-1+p} - z_{\text{S}}). \tag{19}$$

The calculation of equation (19) is carried out by the routine `GetGTwiddle` in LIBSUBSTRATE.

Green's functions for potentials

In equation (18) I am computing the 6 components of the $\mathbf{E}$ and $\mathbf{H}$ fields produced by the induced surface currents. If instead I compute the *potentials* produced by those currents I obtain a slightly different Green's function. Thus, let $\mathbf{A}^{\text{E}}, \Phi^{\text{E}}$ be the usual vector and scalar potential of an electric-current source in a homogeneous region, and let $\mathbf{A}^{\text{M}}, \Phi^{\text{M}}$ be their counterparts for magnetic-current sources, i.e. if the electric and magnetic volume currents are $\mathbf{J}$ and $\mathbf{M}$ then

$$\mathbf{A}^{\text{E}}(\mathbf{x}_{\text{D}}) = \mu \int \mathbf{J}(\mathbf{x}_{\text{S}}) G_0(\mathbf{x}_{\text{DS}})\, d\mathbf{x}_{\text{S}}, \quad \Phi^{\text{E}}(\mathbf{x}_{\text{D}}) = \frac{1}{i\omega\epsilon} \int (\nabla \cdot \mathbf{J})\, G_0(\mathbf{x}_{\text{DS}}) d\mathbf{x}_{\text{S}} \tag{20a}$$

$$\mathbf{A}^{\text{M}}(\mathbf{x}_{\text{D}}) = \epsilon \int \mathbf{M}(\mathbf{x}_{\text{S}}) G_0(\mathbf{x}_{\text{DS}})\, d\mathbf{x}_{\text{S}}, \quad \Phi^{\text{M}}(\mathbf{x}_{\text{D}}) = \frac{1}{i\omega\mu} \int (\nabla \cdot \mathbf{M})\, G_0(\mathbf{x}_{\text{DS}}) d\mathbf{x}_{\text{S}} \tag{20b}$$

with $\mathbf{x}_{\text{DS}} \equiv \mathbf{x}_{\text{D}} - \mathbf{x}_{\text{S}}$ and

$$G_0(k; \mathbf{r}) = \frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|} = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \widetilde{G}_0(\mathbf{q}, z) e^{i\mathbf{q}\cdot\boldsymbol{\rho}}, \qquad \widetilde{G}_0 = \frac{i}{2q_z} e^{iq_z|z|}.$$

I write

2.3 Reduction of 2D Fourier integrals to 1D (Sommerfeld) integrals

The real-space DGF correction is the inverse Fourier transform of (19):

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\rho}, z_{\rm D}, z_{\rm S}) = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q}; z_{\rm D}; z_{\rm S}) e^{i\mathbf{q}\cdot\boldsymbol{\rho}}$$

or, in polar coordinates with $(q_x, q_y) = (q\cos\theta_q, q\sin\theta_q)$, $(\rho_x, \rho_y) = (\rho\cos\theta_\rho, \rho\sin\theta_\rho)$,

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\rho}) = \int_0^\infty \frac{q\,dq}{2\pi} \int_0^{2\pi} \frac{d\theta_q}{2\pi} \widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q}) e^{iq\rho\cos(\theta_q - \theta_\rho)}. \tag{21}$$

(Here and for much of this section I suppress $z_{\rm D,S}$ arguments, but one must remember that they are always there.[5]) The goal of this section is to integrate out the angular variable θ_q to reduce the 2D integral over $\mathbf{q}$ to a 1D integral over $q = |\mathbf{q}|$. In abbreviated form this proceeds as follows:

1. Separate variables by writing $\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q})$ as a sum of products of θ_q-independent scalar functions $\widetilde{g}(q)$ times $q-$independent matrix-valued functions $\boldsymbol{\Lambda}(\theta_q)$ (Section 2.3.1):
$$\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q}) = \sum_{n=1}^{18} \widetilde{g}^{(n)}(q) \boldsymbol{\Lambda}^{(n)}(\theta_q)$$

2. Evaluate integrals over θ_q analytically to yield Bessel functions $J_\nu(q\rho)$ multiplying q-independent matrix-valued functions $\boldsymbol{\Lambda}(\theta_\rho)$ (Section 2.3.2). After this step (21) reads
$$\boldsymbol{\mathcal{G}}(\boldsymbol{\rho}) = \sum_{m=1}^{22} \underbrace{\left[\int_0^\infty \widetilde{\mathfrak{g}}^{(m)}(q,\rho)\,dq\right]}_{\mathfrak{g}^{(m)}(\rho)} \boldsymbol{\Lambda}^{(m)}(\theta_\rho) \tag{22}$$
where the $\widetilde{\mathfrak{g}}(q,\rho)$ functions are linear combinations of the $\widetilde{g}(q)$ functions times Bessel functions in $q\rho$ and other factors.

3. Evaluate the remaining integrals over q numerically using sophisticated tricks for evaluating Sommereld integrals (Section 2.3.3).

2.3.1 Factor $\widetilde{\boldsymbol{\mathcal{G}}}$ into q-independent and $\theta_{\mathbf{q}}$-independent terms

I begin by noting that $\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q})$ may be decomposed as a sum of scalar functions of $q = |\mathbf{q}|$ times q-independent matrix-valued functions of $\theta_{\mathbf{q}}$:

$$\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q}) = \sum_{n=1}^{18} \widetilde{g}^{(n)}(q) \boldsymbol{\Lambda}^{(n)}(\theta_{\mathbf{q}}) \tag{23}$$

[5] More specifically, the "g-like" quantities $\boldsymbol{\mathcal{G}}(\boldsymbol{\rho}), \widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q}), \widetilde{g}(q), \widetilde{\mathfrak{g}}(q,\rho)$, and $\mathfrak{g}(\rho)$ all depend on $z_{\rm S,D}$, but the matrix-valued functions $\boldsymbol{\Lambda}_n(\theta)$ do not.

For example, the upper two quadrants read

$$\widetilde{\boldsymbol{\mathcal{G}}}^{\text{EE}}(\mathbf{q}) = \widetilde{g}^{\text{EE0}\parallel}(q)\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{\Lambda}^{0\parallel}} + \widetilde{g}^{\text{EE0}z}(q)\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{\Lambda}^{0z}}$$

$$+\widetilde{g}^{\text{EE1}}(q)\underbrace{\begin{pmatrix} 0 & 0 & \cos\theta_{\mathbf{q}} \\ 0 & 0 & \sin\theta_{\mathbf{q}} \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{\Lambda}^{1}(\theta_{\mathbf{q}})} + \widetilde{g}^{\text{EE1}\top}(q)\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos\theta_{\mathbf{q}} & \sin\theta_{\mathbf{q}} & 0 \end{pmatrix}}_{\mathbf{\Lambda}^{1\top}(\theta_{\mathbf{q}})}$$

$$+\widetilde{g}^{\text{EE2}}(q)\underbrace{\begin{pmatrix} \cos^2\theta_{\mathbf{q}} & \cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & 0 \\ \cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & \sin^2\theta_{\mathbf{q}} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{\Lambda}^{2}(\theta_{\mathbf{q}})}$$

$$\widetilde{\boldsymbol{\mathcal{G}}}^{\text{EM}}(\mathbf{q}) = \widetilde{g}^{\text{EM0}\parallel}(q)\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{\Lambda}^{0\times}} + \widetilde{g}^{\text{EM2}}(q)\underbrace{\begin{pmatrix} \cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & \sin^2\theta_{\mathbf{q}} & 0 \\ -\cos^2\theta_{\mathbf{q}} & -\cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{\Lambda}^{2\times}}$$

$$+\widetilde{g}^{\text{EM1}}(q)\underbrace{\begin{pmatrix} 0 & 0 & -\sin\theta_{\mathbf{q}} \\ 0 & 0 & +\cos\theta_{\mathbf{q}} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{\Lambda}^{1\times}} + \widetilde{g}^{\text{EM1}\top}(q)\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin\theta_{\mathbf{q}} & \cos\theta_{\mathbf{q}} & 1 \end{pmatrix}}_{\mathbf{\Lambda}^{1\times\top}}$$

where the $\top$ superscript indicates matrix transpose. The expressions for $\widetilde{\boldsymbol{\mathcal{G}}}^{\text{ME}}$ and $\widetilde{\boldsymbol{\mathcal{G}}}^{\text{MM}}$ are similar, involving the same Λ matrices with different $\widetilde{g}$ prefactors.

2.3.2 Evaluate $\theta_{\mathbf{q}}$ integrals

Using Table 3, the $\theta_{\mathbf{q}}$ integral in (21) may be evaluated analytically to yield Bessel-function factors $J_\nu(q\rho)$ ($\nu \in \{0,1,2\}$) times $\mathbf{\Lambda}$ matrices, now evaluated

$$\frac{1}{2\pi}\int_0^{2\pi} e^{iq\rho\cos(\theta_q-\theta_\rho)}\left\{\begin{array}{c} 1 \\ \cos\theta_q \\ \sin\theta_q \\ \cos^2\theta_q \\ \cos\theta_q\sin\theta_q \\ \sin^2\theta_q \end{array}\right\} d\theta_q = \left\{\begin{array}{l} J_0(q\rho) \\ iJ_1(q\rho)\cos\theta_\rho \\ iJ_1(q\rho)\sin\theta_\rho \\ -J_2(q\rho)\cos^2\theta_\rho + \frac{J_1(q\rho)}{q\rho} \\ -J_2(q\rho)\cos\theta_\rho\sin\theta_\rho \\ -J_2(q\rho)\sin^2\theta_\rho + \frac{J_1(q\rho)}{q\rho} \end{array}\right\},$$

Figure 3: Table of integrals used to reduce 2D integrals over $\mathbf{q}$ to 1D integrals over $|q|$.

at $\theta_{\boldsymbol{\rho}}$. For example, one term in the expansion of $\boldsymbol{\mathcal{G}}(\boldsymbol{\rho})$ is

$$\begin{aligned}\int_0^\infty \frac{qdq}{2\pi}\widetilde{g}^{\text{EE1}}(q)\underbrace{\int_0^{2\pi}\frac{d\theta_q}{2\pi}\mathbf{\Lambda}^1(\theta_q)e^{iq\rho\cos(\theta_q-\theta_\rho)}}_{iJ_1(q\rho)\mathbf{\Lambda}^1(\theta_\rho)} \\ = \underbrace{\left\{\int_0^\infty dq\underbrace{\left[\frac{q}{2\pi}\widetilde{g}^{\text{EE1}}(q)\cdot iJ_1(q\rho)\right]}_{\widetilde{\mathfrak{g}}^{\text{EE1}}(q,\rho)}\right\}}_{\mathfrak{g}^{\text{EE1}}(\rho)}\mathbf{\Lambda}^1(\theta_\rho)\end{aligned}$$

The second line here defines some new symbols: $\widetilde{\mathfrak{g}}$ are functions of q and ρ defined as products of $\widetilde{g}(q)$ factors times $J_\nu(q\rho)$ factors and other factors, while $\mathfrak{g}$ are functions of ρ obtained by integrating out the q dependence of $\mathfrak{g}(q,\rho)$. The full set of rules defining the $\widetilde{\mathfrak{g}}$ is

$$\widetilde{\mathfrak{g}}^{\text{EE0}\parallel}(q,\rho) \equiv \frac{q}{2\pi}\left[\widetilde{g}^{\text{EE0}\parallel}(q)J_0(q\rho) + \widetilde{g}^{\text{EE2}}(q)\frac{J_1(q\rho)}{q\rho}\right] \tag{24a}$$

$$\widetilde{\mathfrak{g}}^{\text{EE0z}}(q,\rho) \equiv \frac{q}{2\pi}\widetilde{g}^{\text{EE0z}}(q)J_0(q\rho) \tag{24b}$$

$$\widetilde{\mathfrak{g}}^{\text{EE1}}(q,\rho) \equiv i\frac{q}{2\pi}\widetilde{g}^{\text{EE1}}(q)J_1(q\rho) \tag{24c}$$

$$\widetilde{\mathfrak{g}}^{\text{EE1}\top}(q,\rho) \equiv i\frac{q}{2\pi}\widetilde{g}^{\text{EE1}\top}(q)J_1(q\rho) \tag{24d}$$

$$\widetilde{\mathfrak{g}}^{\text{EE2}}(q,\rho) \equiv -\frac{q}{2\pi}\widetilde{g}^{\text{EE2}}(q)J_2(q\rho) \tag{24e}$$

$$\widetilde{\mathfrak{g}}^{\text{EM0}\parallel\,\times}(q,\rho) \equiv \frac{q}{2\pi}\left[\widetilde{g}^{\text{EM0}\parallel}(q)J_0(q\rho) + \widetilde{g}^{\text{EM2}}(q)\frac{J_1(q\rho)}{q\rho}\right] \tag{24f}$$

$$\widetilde{\mathfrak{g}}^{\text{EM1}\times}(q,\rho) \equiv i\frac{q}{2\pi}\widetilde{g}^{\text{EM1A}}(q)J_1(q\rho) \tag{24g}$$

$$\widetilde{\mathfrak{g}}^{\text{EM1}\times\top}(q,\rho) \equiv i\frac{q}{2\pi}\widetilde{g}^{\text{EM1B}}(q)J_1(q\rho) \tag{24h}$$

$$\widetilde{\mathfrak{g}}^{\text{EM2}\times}(q,\rho) \equiv -\frac{q}{2\pi}\widetilde{g}^{\text{EM2}}J_2(q\rho) \tag{24i}$$

2.3.3 Evaluate Sommerfeld integrals over q

Assembling the above pieces, the substrate DGF correction $\boldsymbol{\mathcal{G}}$ is a sum of 22 terms:[6]

$$\boldsymbol{\mathcal{G}}(\rho) = \sum_{m=1}^{22} \mathfrak{g}^{(m)}(\rho)\boldsymbol{\Lambda}^{(m)}(\theta_\rho),$$

where the $\mathfrak{g}^{(m)}(\rho)$ functions are defined by Sommerfeld integrals:

$$\mathfrak{g}^{(m)}(\rho) \equiv \int_0^\infty \widetilde{\mathfrak{g}}^{(m)}(q,\rho)\,dq. \tag{25}$$

[6]This tally treats the integrals of the two integrand terms on the RHS of (24a) as two separate integrals [and similarly for (24f) and the corresponding equations for the ME and MM quadrants]. If the terms are lumped together then the number of distinct $\mathfrak{g}$ functions is 18.

3 SCUFF-EM integration: Substrate contributions to BEM matrix and RHS vector

3.1 Fields of individual basis functions

$$\boldsymbol{\mathcal{G}}^{\text{EE}} =$$

3.2 SIE matrix elements: Panel-panel integrals

If $\mathcal{S}_\alpha, S_\beta$ are two `RWGSurfaces` exposed to the outermost (ambient) region in a SCUFF-EM geometry, then the elements of the SIE matrix elements corresponding to any pair of basis functions $\{\mathbf{b}_a \in \mathcal{S}_\alpha, \mathbf{b}_b \in \mathcal{S}_\beta\}$ receive corrections of the form

$$\begin{aligned}\Delta M_{ab}^{\text{PQ}} &= \Big\langle \mathbf{b}_a \Big| \boldsymbol{\mathcal{G}}^{\text{PQ}} \Big| \mathbf{b}_b \Big\rangle \\ &\equiv \iint \mathbf{b}_a(\mathbf{x}_a) \cdot \boldsymbol{\mathcal{G}}^{\text{PQ}}(\mathbf{x}_a, \mathbf{x}_b) \cdot \mathbf{b}_b(\mathbf{x}_b)\, d\mathbf{x}_b\, d\mathbf{x}_a \qquad (26)\end{aligned}$$

I will consider two different approaches for evaluating the panel-panel integrals[7] here:

1. The *spectral inner* approach: In this case I simply evaluate the panel-panel cubature in (26), with values of $\boldsymbol{\mathcal{G}}$ at each cubature point computed via the methods of LIBSUBSTRATE as described in the previous section (possibly accelerated via interpolation tables). I call this the "spectral inner" method because in this case the q integral in the definition of $\boldsymbol{\mathcal{G}}$ is the innermost of 3 integrals. Indeed, inserting equation (22) we have

$$\Delta M_{ab}^{\text{PQ}} \equiv \iint \mathbf{b}_a(\mathbf{x}_a) \Big\{ \sum \mathfrak{g}^{(m)}(\rho) \boldsymbol{\Lambda}^{(m)}(\theta_\rho) \Big\} \mathbf{b}_b(\mathbf{x}_b)\, d\mathbf{x}_b\, d\mathbf{x}_a$$

[where $\boldsymbol{\rho} = (\mathbf{x}_a - \mathbf{x}_b)_\parallel = (\rho\cos\theta_\rho, \rho\sin\theta_\rho)$]. Recalling the definition (25), this is a sum of triple integrals:

$$\equiv \iint \mathbf{b}_a(\mathbf{x}_a) \left\{ \sum \left[\int_0^\infty \widetilde{\mathfrak{g}}^{(m)}(q,\rho) dq \right] \boldsymbol{\Lambda}^{(m)}(\theta_\rho) \right\} \cdot \mathbf{b}_b(\mathbf{x}_b)\, d\mathbf{x}_b\, d\mathbf{x}_a. \qquad (27)$$

2. The *spectral outer* approach: In this case I rearrange the order of integration in (28) so that the q integral is the *outermost* integral, with an integrand defined for each q by a panel-panel integral involving the spectral-domain GF:

$$\Delta M_{ab}^{\text{PQ}} = \int_0^\infty \left\{ \iint \mathbf{b}_a(\mathbf{x}_a) \Big[\sum \widetilde{\mathfrak{g}}^{(m)}(q,\rho) \boldsymbol{\Lambda}^{(m)}(\theta_\rho) \Big] \mathbf{b}_b(\mathbf{x}_b)\, d\mathbf{x}_b\, d\mathbf{x}_a \right\} dq \qquad (28)$$

[7] I refer to 4-dimensional integrals like (26) as "panel-panel integrals" because they are a sum of contributions of integrals over pairs of flat triangular panels.

$$\mathcal{G}_{ij}^{\text{EE}} = \delta_{ij}$$

4 Metal-on-Insulator geometries

$$\begin{aligned}\mathbf{E} &= iw\mathbf{A} - \nabla\phi \\ &= iw\mu G_0 \star \mathbf{J} - \frac{1}{i\omega\epsilon}\nabla G_0 \star \boldsymbol{\rho}\end{aligned}$$

$$\begin{aligned}\widetilde{G}^{\mathrm{App}} &= \frac{1}{2\pi} q J_0(q\rho)\zeta^{\mathrm{App}} \\ \widetilde{G}^{\mathrm{Apz}} &= -\frac{1}{2\pi}(\epsilon_r - 1)q^2 J_1(q\rho)\zeta^{\mathrm{Apz}} \\ \widetilde{G}^{\Phi} &= \frac{1}{2\pi}\rho J_0(q\rho)\zeta^{\Phi}\end{aligned}$$

$$\begin{aligned}\zeta^{\mathrm{App}} &= \frac{1}{D^{\mathrm{TE}}} \times \begin{cases} e^{-u_0 z} \\ \frac{\sinh u(z+h)}{\sinh uh} \end{cases} \\ \zeta^{\mathrm{Apz}} &= \frac{1}{D^{\mathrm{TE}} D^{\mathrm{TM}}} \times \begin{cases} e^{-u_0 z} \\ \frac{\cosh u(z+h)}{\cosh uh} \end{cases} \\ \zeta^{\Phi} &= \frac{N}{D^{\mathrm{TE}} D^{\mathrm{TM}}} \times \begin{cases} e^{-u_0 z} \\ \frac{\sinh u(z+h)}{\sinh uh} \end{cases}\end{aligned}$$

$$\frac{N}{D_{\mathrm{TE}} D_{\mathrm{TM}}} \xrightarrow{u \to u_0} \frac{1 - e^{-2u_0 h}}{u_0(\epsilon + 1)} \sum_{n=0}^{\infty} \Big[- \eta e^{-2u_0 h} \Big]^n$$

5 Unit-test framework

The LIBSUBSTRATE standalone library comes with a unit-test suite to test core functionality related to calculation of substrate DGFs. Separately, the unit-test suite for LIBSCUFF includes tests to check the integration of LIBSUBSTRATE into LIBSCUFF.

5.1 LIBSUBSTRATE unit tests

5.1.1 `tGTwiddle`

The unit-test code `tGTwiddle.cc` tests that the full Fourier-space DGF $\widetilde{\mathbf{\Gamma}}(\mathbf{q}, z_{\text{D}}, z_{\text{S}})$ satisfies the appropriate boundary conditions at each layer of the layered substrate, namely

$$C^{+}(\text{P}, i, \ell)\widetilde{\Gamma}^{\text{PQ}}_{ij}(\mathbf{q}, z_\ell + \eta, z_{\text{S}})C^{-}(\text{P}, i, \ell)\widetilde{\Gamma}^{\text{PQ}}_{ij}(\mathbf{q}, z_\ell - \eta, z_{\text{S}}) \tag{29}$$

where

$$C^{\pm}(P, i, \ell) = \begin{cases} 1, & i \in \{x, y\} \\ \epsilon^{\pm}_\ell, & i = z, P = E \\ \mu^{\pm}_\ell, & i = z, P = H \end{cases}$$

where $\{\epsilon, \mu\}^{\pm}_\ell$ are the material properties for the layer above/below z_ℓ, i.e. (Figure **??**)

$$\{\epsilon_\ell, \mu_\ell\}^{+} = \{\epsilon_\ell, \mu_\ell\}, \qquad \{\epsilon_\ell, \mu_\ell\}^{-} = \{\epsilon_{\ell+1}, \mu_{\ell+1}\}.$$

If a ground plane is present, we have the additional condition

$$\widetilde{\Gamma}^{\text{PQ}}_{ij}(q, z_{\text{GP}}, z_{\text{S}}) = 0 \quad \text{for} \quad i \in \{x, y\}. \tag{30}$$

Conditions (29) and (30) must hold *independently* of the indices $\text{Q} \in \{\text{E}, \text{H}\}$ and $j \in \{1, 2, 3\}$ and of the values of $\mathbf{q}$ and z_{S}.

A Symbols and indices used in this document

A.1 Symbols

Symbol	Arguments	Description
$\boldsymbol{\mathcal{F}}$	$\mathbf{r}$, geometry	Field six-vector $\boldsymbol{\mathcal{F}} = \binom{\mathbf{E}}{\mathbf{H}}$
$\boldsymbol{\mathcal{C}}$	$\mathbf{r}$, geometry	Current six-vector $\boldsymbol{\mathcal{C}} = \binom{\mathbf{J}}{\mathbf{M}}$ or $\boldsymbol{\mathcal{C}} = \binom{\mathbf{K}}{\mathbf{N}}$
$\boldsymbol{\Gamma}$	$\boldsymbol{\rho}, z_{\text{D}}, z_{\text{S}}, \omega$, geometry	Full (bare+scattered) 6×6 dyadic Green's function, $\boldsymbol{\mathcal{F}} = \boldsymbol{\Gamma} \star \boldsymbol{\mathcal{C}}$
$\boldsymbol{\Gamma}^{0r}$	$\boldsymbol{\rho}, z_{\text{D}}, z_{\text{S}}, \omega, \epsilon^r, \mu^r$	Bare (homogeneous) 6×6 dyadic Green's function in region r
$\boldsymbol{\mathcal{G}}$	$\boldsymbol{\rho}, z_{\text{D}}, z_{\text{S}}, \omega$, geometry	Scattering contribution to $\boldsymbol{\Gamma}$ ($\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{0r} + \boldsymbol{\mathcal{G}}$)
$\boldsymbol{\mathcal{P}}$	$\mathbf{r}$, geometry	Potential eight-vector $\boldsymbol{\mathcal{P}} = \begin{pmatrix} \mathbf{A}^{\text{E}} \\ \Phi^{\text{E}} \\ \mathbf{A}^{\text{M}} \\ \Phi^{\text{M}} \end{pmatrix}$
$\boldsymbol{\mathcal{S}}$	$\mathbf{r}$, geometry	Source eight-vector $\boldsymbol{\mathcal{S}} = \begin{pmatrix} \mathbf{J} \\ \rho^{\text{E}} \\ \mathbf{M} \\ \rho^{\text{M}} \end{pmatrix}$
$\boldsymbol{\Lambda}$	$\boldsymbol{\rho}, z_{\text{D}}, z_{\text{S}}, \omega$, geometry	Full (bare+scattered) 8×8 dyadic Green's function, $\boldsymbol{\mathcal{P}} = \boldsymbol{\Pi} \star \boldsymbol{\mathcal{S}}$
$\boldsymbol{\Lambda}^{0r}$	$\boldsymbol{\rho}, z_{\text{D}}, z_{\text{S}}, \omega$, geometry	Bare (homogeneous) 8×8 dyadic Green's function for region r
$\boldsymbol{\mathcal{L}}$	$\boldsymbol{\rho}, z_{\text{D}}, z_{\text{S}}, \omega$, geometry	Scattering contribution to $\boldsymbol{\Lambda}$ ($\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{0r} + \boldsymbol{\mathcal{L}}$)

A.2 Indices

<table>
<tr><th>Index</th><th>Range</th><th>Significance</th></tr>
<tr><td>i, j</td><td>$\{1, 2, 3\}$</td><td>Cartesian directions x, y, z</td></tr>
<tr><td>I, J</td><td>$\{1, 2, 3, 4, 5, 6\}$</td><td>Electric/magnetic field/current components
<table><tr><td>1,2,3</td><td>$E_{x,y,z}$, $J_{x,y,z}$, $K_{x,y,z}$</td></tr><tr><td>4,5,6</td><td>$H_{x,y,z}$, $M_{x,y,z}$, $N_{x,y,z}$</td></tr></table></td></tr>
<tr><td>μ, ν</td><td>$\{1, 2, 3, 4, 5, 6, 7, 8\}$</td><td>Electric/magnetic potential/source components
<table><tr><td>1,2,3</td><td>$A^{\mathrm{E}}_{x,y,z}$, $J_{x,y,z}$, $K_{x,y,z}$</td></tr><tr><td>4</td><td>Φ^{E}, ρ^{E}, σ^{E}</td></tr><tr><td>5,6,7</td><td>$A^{\mathrm{M}}_{x,y,z}$, $M_{x,y,z}$, $N_{x,y,z}$</td></tr><tr><td>8</td><td>Φ^{M}, ρ^{M}, σ^{M}</td></tr></table></td></tr>
</table>

B 8×8 Dyadic Green's Functions

The usual 6×6 dyadic Green's function $\mathbf{\Gamma}$ operates on a six-vector of currents to yield a six-vector of fields. It is convenient to consider a slightly different object that operates on an *eight*-vector of sources to yield an *eight*-vector of potentials.

In the presence of magnetic currents, the usual (electric-current-sourced) vector and scalar potentials $\mathbf{A}^{\text{E}}, \Phi^{\text{E}}$, are joined by their magnetic-current-sourced counterparts $\mathbf{A}^{\text{M}}, \Phi^{\text{M}}$, which are related to the fields according to

$$\begin{aligned}\mathbf{E} &= i\omega\mu\mathbf{A}^{\text{E}} - \frac{1}{i\omega\epsilon}\nabla\Phi^{\text{E}} - \nabla\times\mathbf{A}^{\text{M}}\\ \mathbf{M} &= \nabla\times\mathbf{A}^{\text{E}} + i\omega\epsilon\mathbf{A}^{\text{M}} - \frac{1}{i\omega\mu}\nabla\Phi^{\text{M}}.\end{aligned}$$

In a homogeneous region, the potentials[8] produced by given source distributions $\{\mathbf{J}, \mathbf{M}\}$ are

$$\begin{aligned}\mathbf{A}^{\text{E}}(\mathbf{x}_{\text{D}}) &= \int G_0(\mathbf{x}_{\text{D}} - \mathbf{x}_{\text{S}})\mathbf{J}(\mathbf{x}_{\text{S}})\, d\mathbf{x}_{\text{S}}, &\qquad \Phi^{\text{E}}(\mathbf{x}_{\text{D}}) &= \int G_0(\mathbf{x}_{\text{D}} - \mathbf{x}_{\text{S}})\big[\nabla\cdot\mathbf{J}\big]\, d\mathbf{x}_{\text{S}}\\ \mathbf{A}^{\text{M}}(\mathbf{x}_{\text{D}}) &= \int G_0(\mathbf{x}_{\text{D}} - \mathbf{x}_{\text{S}})\mathbf{M}(\mathbf{x}_{\text{S}})\, d\mathbf{x}_{\text{S}}, &\qquad \Phi^{\text{M}}(\mathbf{x}_{\text{D}}) &= \int G_0(\mathbf{x}_{\text{D}} - \mathbf{x}_{\text{S}})\big[\nabla\cdot\mathbf{M}\big]\, d\mathbf{x}_{\text{S}}\end{aligned}$$

where

$$G_0(\mathbf{r}) = \frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|}.$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} i\omega\mu G_0 & 0 & 0 & -\frac{1}{i\omega\epsilon}\partial_x G_0 & 0 & \partial_z G_0 & -\partial_y G_0 & 0 \\ 0 & i\omega\mu G_0 & 0 & -\frac{1}{i\omega\epsilon}\partial_y G_0 & -\partial_z G_0 & 0 & \partial_x G_0 & 0 \\ 0 & 0 & i\omega\mu G_0 & -\frac{1}{i\omega\epsilon}\partial_z G_0 & \partial_y G_0 & -\partial_x G_0 & 0 & 0 \\ 0 & -\partial_z G_0 & \partial_y G_0 & 0 & i\omega\epsilon G_0 & 0 & 0 & -\frac{1}{i\omega\mu}\partial_x G_0 \\ \partial_z G_0 & 0 & \partial_x G_0 & 0 & 0 & i\omega\epsilon G_0 & 0 & -\frac{1}{i\omega\mu}\partial_y G_0 \\ -\partial_y G_0 & \partial_x G_0 & 0 & 0 & 0 & 0 & i\omega\epsilon G_0 & -\frac{1}{i\omega\mu}\partial_z G_0 \end{pmatrix} \star \begin{pmatrix} J_x \\ J_y \\ J_z \\ \nabla\cdot\mathbf{J} \\ M_x \\ M_y \\ M_z \\ \nabla\cdot\mathbf{M} \end{pmatrix}$$

[8]Note that my $\Phi^{\text{E,M}}$ are $i\omega$ times the actual scalar potentials due to the charge distributions associated with currents $\mathbf{J}, \mathbf{M}$.